# How to estimate Seshadri constants



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## 1 Seshadri constant

For an ample line bundle, sometimes we want to know how positive it is. So, we define an invariant which measures the positivities of line bundles.

$X$ : projective algebraic variety over $\mathbb{C}$
$L$ : ample line bundle on $X$
$p$ : closed point on $X$

We define Seshadri constant $\varepsilon(p; X, L)$ as follows;

$$\varepsilon(p; X, L) = \varepsilon(p; L) := \inf\left\{\frac{C.L}{\mathrm{mult}_p(C)} | C \subset X; \text{curve containing } p\right\}.$$

Note Seshadri's criterion for amplitude states that a divisor $D$ on $X$ is ample if and only if

$$\inf_{p \in X} \inf_{p \in C \subset X} \left\{\frac{C.D}{\mathrm{mult}_p(C)}\right\} > 0.$$

So we can consider that Seshadri constants $\varepsilon(p; L)$ measure the positivity of $L$ at $p$.

**Example 1.1.** (1) $\varepsilon(p; \mathbb{P}^n, \mathcal{O}(k)) = k$ for $p \in \mathbb{P}^n$.
(2) $\varepsilon(p; \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b)) = \min\{a, b\}$ for $p \in \mathbb{P}^1 \times \mathbb{P}^1$.

It is easily shown that

$$\varepsilon(p; L) = \max\{s \in \mathbb{R} | \ \mu^*(L) - sE \text{ is nef }\}$$

where $\mu : \widetilde{X} \to X$ is the blow-up at $p$ and $E := \mu^{-1}(p)$ is the exceptional divisor. So there is an upper bound $\varepsilon(p; L) \leq \sqrt[n]{L^n}$ for $n = \dim X$.

Seshadri constants relate with adjoint bundles, slope stabilities, an invariant of symplectic manifolds and so on. So we want to compute or estimate them, but unfortunately, it is difficult in general.

## 2 Toric cases

Let $M$ be an abelian group of rank $n \in \mathbb{N}$ and set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For a closed convex set $\Delta$ in $M_{\mathbb{R}}$, we define real numbers $s(\Delta), s'(\Delta)$ as follows;

At first, we define $s_{e_1,\ldots,e_n}(\Delta) \in [0, \infty]$ by induction of $n$ for a basis $(e_1, \ldots, e_n)$ of $M$.

In case of $n = 1$, $s_{e_1}(\Delta) := |\Delta|$, the length of $\Delta$.

If we can define for $1, \ldots, n-1$, then we set

$$s_{e_1,\ldots,e_n}(\Delta) := \min\left\{|p_n(\Delta)|, \sup_{t \in \mathbb{R}}\{s_{e_1,\ldots,e_{n-1}}({p_n}^{-1}(t) \cap \Delta)\}\right\},$$

where $p_n : M_{\mathbb{R}} \to \mathbb{R}$ is the n-th projection with respect to $(e_1, \ldots, e_n)$. Note that ${p_n}^{-1}(t) \cap \Delta$ is regarded as a closed convex subset in $(\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n-1}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Now $s(\Delta), s'(\Delta) \in [0, \infty]$ are defined to be

$$s(\Delta) := \sup_{(e_1,\ldots,e_n):basis}\{s_{e_1,\ldots,e_n}(\Delta)\},$$

$$s'(P) := \inf_{(e_1,\ldots,e_n):basis}\{|p_n(P)|\}.$$

**Example 2.1.** For $\Delta = \triangle OPQ$ in Figure 1 and $e_1 = (1, 0), e_2 = (0, 1)$, $s_{e_1 e_2}(\Delta) = \min\{|RO|, |PS|\} = \frac{3}{2}$, $|p_2(\Delta)| = |RO| = 2$. And in this case we can show $s(\Delta) = \frac{3}{2}$, $s'(\Delta) = 2$.

For an integral polytope $\Delta \subset M_{\mathbb{R}}$, there is a polarized toric variety $(X_\Delta, L_\Delta)$ corresponding to $\Delta$ and we can estimate Seshadri constant of $(X_\Delta, L_\Delta)$ using $s(\Delta), s'(\Delta)$.

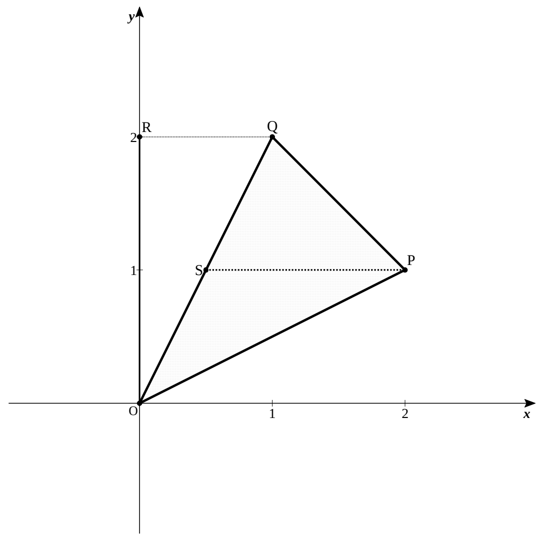


Figure 1: polytope $\triangle OPQ$

**Theorem 2.2.** *For above notations and any point p in the maximal orbit of $X_\Delta$,*

$$s(\Delta) \leq \varepsilon(p; X_\Delta, L_\Delta) \leq s'(\Delta).$$

**Example 2.3.** For $\Delta = \triangle OPQ$, $X = X_\Delta$ is a cubic surface ${X_0}^3 = X_1 X_2 X_3 \subset \mathbb{P}^3$ and $L_\Delta = \mathcal{O}_X(1)$. In this case $s(\Delta) = \frac{3}{2}, s'(\Delta) = 2$ and in fact $\varepsilon(p; \mathcal{O}_X(1)) = \frac{3}{2}$.

## 3 non-toric cases

There is a semicontinuity about Seshadri constants, so we can estimate in some non-toric cases using degenerations and Theorem 2.2.

**Theorem 3.1.** *Let $\{(X_t, L_t)\}_t$ be a flat family of polarized varieties such that it's central fiber $(X_0, L_0)$ is the polarized toric variety $(X_\Delta, L_\Delta)$ for an integral polytope $\Delta$. Then*

$$\varepsilon(p; X_t, L_t) \geq s(\Delta)$$

*for very general $t$ and $p \in X_t$.*

**Example 3.2.** (1) Smooth cubic surfaces in $\mathbb{P}^3$ degenerate to the singular cubic surface ${X_0}^3 = X_1 X_2 X_3 \subset \mathbb{P}^3$, so for very general cubic surface $X$ and very general $p \in X$,

$$\varepsilon(p; X, \mathcal{O}_X(1)) \geq s(\triangle OPQ) = \frac{3}{2}.$$

(2) More generally, complete intersections in $\mathbb{P}^N$ degenerate to toric varieties, so we can estimate Seshadri constants of them.
(3) Spherical varieties degenerate to toric varieties.

## 4 Questions

**Question 4.1.** *As Example 2.3, the lower bound $s(\Delta)$ is not bad. But the upper bound $s'(\Delta)$ is not good. So are there better upper bounds?*

**Question 4.2.** *For a polarized variety, does it degenerate to a polarized toric variety?*

**Question 4.3.** *For a polarized variety of dimension n, we can construct Okounkov bodies, which are convex sets in $\mathbb{R}^n$. They are generalizations of moment polytopes of toric varieties in some sense, so do inequalities as Theorem 2.2 holds about Okounkov bodies?*